Optimal Progressivity of Public Pension Benefit and Labor Income Tax

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Abstract

This paper characterizes optimal progressivity of public pension benefit and labor income tax, which leads to intra- and inter-generational redistributions. The optimal progressivity of public pension benefit and that of labor income tax mutually depend on each other. The optimal ratio of public pension benefit progressivity to labor income tax progressivity is not constant rate of time discounting. Optimal public pension benefit is more progressive than optimal labor income tax. Whereas effects of pre-government inequality on the optimal progressivity of public pension benefit and that of labor income tax are positive, effect of population aging on them is theoretically ambiguous.

1. Introduction

Progressive public pension benefit can alleviate income inequality. For any given degree of income tax progressivity, more progressive public pension benefit reduces income inequality intra- and inter- generations. In fact, the average public pension spending of the OECD economies, over 1980-2015, amounts to 6.64% of GDP, which is significant and fairly similar to their personal income tax revenue (8.88% of GDP, on average over the same period). Furthermore, facing rapidly aging population, progressivity of public pension benefit becomes increasingly important. Despite its importance, so far, progressivity of public pension benefit has not been studied well. Thus, this paper analyzes optimal progressivity of public pension benefit.

In an overlapping-generations model with pay-as-you-go public pension, optimal degree of progressivity of public pension benefit and labor income tax is characterized. Public pen-

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sion benefit progressivity is complementary to labor income tax progressivity for reducing intra- and inter-generational income inequality of workers and retirees. While more progressive income tax decreases the current return on labor supply, more progressive public pension benefit decreases the future return on labor supply. The efficiency loss of labor-supply distortion is traded off for equity enhancement. The optimal ratio of the degree of public pension benefit progressivity to that of labor income tax progressivity is not constant as present-value discounting factor. Rather, it varies depending on the chosen degree of progressivity, the pre-government inequality, and the elderly population share. Importantly, this paper shows that optimal public pension benefit is more progressive than optimal labor income tax. To achieve equity improvement, an increase in the degree of public pension benefit progressivity reduces labor supply by a smaller margin than the same-size increase in the degree of income tax progressivity does, because the former affects the future return of labor supply whereas the latter does the current return. In addition, this paper also shows that whereas an increase in the level of the pre-government inequality clearly raises the optimal degree of progressivity of public pension benefit and that of labor income tax, it is a priori ambiguous whether an increase in the elderly population share (i. e., population aging) does so or not.

This paper is relatable to the literature on optimal public pension benefit. Diamond and Mirrlees (1978 and 1986) first analyzed optimal public pension in terms of consumption and labor-supply paths of ex-ante identical individuals with no explicit variable of public pension benefit. While having an explicit variable of public pension benefit, optimal *linear* rate of public pension benefit is characterized by Feldstein (1985), İmrohoroglu *et al.* (1995), Yew and Zhang (2009), Cremer and Pestieau (2011) and the like. While progressivity of public pension benefit is identified with the slope of the entire schedule of *marginal* rates of public pension benefit, linear rate of public pension benefit does not allow marginal rates to vary but sets the slope fixed at zero. Thus, these studies did not discuss progressivity of public pension benefit. However, actual public pension benefit schedule in many advanced economies is not linear.

In this regard, Huggett and Ventura (1999) and Gustman and Steinmeier (2001) conducted numerical analyses using actual nonlinear formula of public pension benefit of the US, without a theoretical analysis on optimal degree of progressivity of public pension benefit. On the other hand, assuming that public pension formula is the combination of linear (earning-related) rate of pension benefit and the flat amount of minimum guarantee pension, Fehr *et al.* (2013) and that income tax is linear, Fehr and Uhde (2013) and Kudrna *et al.* (2022) carried out numerical simulations and discussed progressivity of public pension. While these studies paid attention to public pension's progressivity, they equated the level of the minimum guarantee pension with the degree of public pension progressivity. However, their assumption on public pension benefit schedule restricts the slope of public pension benefit rates to be same for all individuals who are poor enough to get the minimum guarantee pension, although the minimum guarantee pension serves only for addressing old-age poverty problem rather than entire distribution of pensioners' incomes. Rigorously speaking, old-age poverty reduction alone is not fully representing the progressivity of the entire schedule of public pension benefit, regardless of the minimum guarantee pension is means-tested or not.

In reality, public pension benefit rates are not set to be same but vary by the pre-retirement contributions (which is proportion to labor income) of pensioners in most of advanced economies. In addition, these simulation studies assumed that labor income tax is linear, although actual schedule of labor income tax is nonlinear in most of economies.

Because public pension benefit depends crucially on pre-retirement labor supply that is greatly affected by labor income tax, incorporation of nonlinear labor income tax is necessary for properly analyzing how the government should design progressivity of public pension benefit for addressing lifetime income inequality between the rich (high earning-ability individuals) and the poor (low earning-ability individuals). In this regard, using Mirrlees model (Mirrlees, 1971) that was widely adopted for optimal nonlinear rates of income tax, Cremer *et al.* (2004) theoretically analyzed optimal nonlinear rates of public pension benefit and labor income tax by imposing unusual assumptions of zero interest rate (no savings) and no time preference. Furthermore, using version of Mirrlees model (Golosov *et al.*, 2006), Golosov *et al*. (2016) relaxed the assumptions of zero interest rate and theoretically characterized optimal nonlinear rates of social insurance and income tax. Although these two studies allowed the government to determine nonlinear rates of public pension benefit and labor income tax together, none of them was able to analyze progressivity of public pension benefit, because both failed to identify optimal marginal rate of public pension benefit separate from that of labor income tax. Due to the nature of Mirrlees model, these two studies identified *only* optimal marginal rate of substitution of consumption and leisure which is equated with one minus labor distortion (marginal labor income tax rate net of the present-value marginal public pension benefit rate). With one equation for optimal labor distortion under Mirrlees model, the two unknowns of marginal public pension benefit rate and marginal labor income tax rate are not uniquely identifiable. As an obvious consequence, it is impossible for Cremer *et al.* (2004) and Golosov *et al.* (2016) to identify and analyze progressivity of public pension benefit itself.[1] In reality, schedule of nonlinear public pension benefit rates is set separately from schedule of nonlinear labor income tax rates.

Distinct from and related to the above-noted studies, with an explicit variable of nonlinear public pension benefit separated from nonlinear income tax, this paper makes novel contributions by offering a theoretical analysis on the progressivity of the entire *nonlinear* schedules of public pension benefit and labor income tax respectively. Obtaining optimal nonlinear rates of public pension benefit and labor income tax and optimal progressivity of them respectively is unique contribution as well as has useful policy implication. In addition, while the results of simulation studies are inevitably limited to a specific set of data values and functional choices, this paper improves the general applicability by finding theoretical results that are not confined to specific data values

This paper is organized as follows. Section 2 describes the model, from which optimal progressivity of public pension benefit and labor income tax is characterized in Section 3. Based on this, the effect of pre-government inequality and population aging, respectively, on the optimal progressivity of public pension benefit and labor income tax is analyzed in Section 4. Section 5 concludes the paper.

2. The Economic Environment

2.1. Individuals

Consider a small open steady-state economy that is populated by a continuum of individuals of two generations (workers and retirees). Each individual lives up to two periods, working for the first period and then being retired for the second period. An individual worker survives to be a retiree with the probability of $\frac{\delta}{1-\delta} \in (0,1)$. Every period, workers of size $1-\delta$ are newly born into this economy with no capital endowment, while individual workers are endowed with different earning abilities that do not change for life. The population size of this economy stays as one. The population share of retirees is $\delta \in (0,1)$ while that of workers is $1-\delta$. Moreover, earning ability $\theta \in [1,\infty)$ is distributed following a Pareto distribution of *Pareto*(α) with $\alpha > 1$ and the cumulative distribution function (CDF) of F_θ. As well known, Pareto distribution is the most widely adopted in the optimal income taxation literature. Notice that the parameter of α indicates pre-government inequality. The utility function[2] of each worker is

$$U_W = log(c_W) - \frac{l^{1+\eta}}{1+\eta} + \chi \, log(G) + \beta E[log(c_R) + \chi \, log(G)] \tag{1}$$

while that of each retiree is

$$U_R = log(c_R) + \chi \, log(G) \tag{2}$$

where $c_W > 0$ and $c_R > 0$ are pre- and post-retirement consumption, respectively; $l \in (0,1)$ is labor supplied by a worker; $\frac{1}{\eta} > 0$ is Frisch labor-supply elasticity; $G > 0$ is public goods provided by the government; $\chi > 0$ is relative preference for public goods; $\beta \in (0,1)$ is time preference. When maximizing (1), an individual worker faces the following inter-temporal budget constraint.

$$\{y - T(y)\} + \frac{1}{(1+r)} E[B(y)] \geq c_W + \frac{1}{(1+r)} E[c_R] \tag{3}$$

where y is pre-tax labor income $y = \theta l$; $T(y)$ is sum of labor income tax and public pension contribution, both of which depend on labor income; $r > 0$ is interest rate; and, $B(y)$ is public pension benefit that is receivable after retirement and depends on pre-tax labor income because it is based on public pension contribution before retirement. In particular, $T(y)$ is decomposed into the following two parts: public pension contribution of $\rho y(\theta)$ and the labor income tax payment of $T(y(\theta)) - \rho y(\theta)$. The former finances public pension benefits for the current retirees, whereas the latter finances public good provision for all. Moreover, expenditure for public pension benefits is independent of public good provision expenditure. Notably, $\rho \in (0,1)$ reflects the reality that public pension contribution rate is flat in most of advanced economies, like the FICA tax rate of the US. Reflecting the fact that firms in some

countries pay public pension contributions as a part of compensation for their employees, a part of $\rho y(\theta)$ can be paid by the firm on behalf its individual employee, without affecting market equilibrium wage. Both public pension benefit and post-retirement consumption are not exactly certain for workers, because it is uncertain for workers to survive the second period. On the other hand, public pension benefit formula of the government certainly determines $B(y)$; and, the rate of return on first-period savings is certainly r. When maximizing (2), an individual retiree faces the following budget constraint:

$$B(y) + (1 + r)k \geq c_R \tag{4}$$

where k is savings made in the first period. Most of the previous studies assumed that income tax is not levied on public pension benefit (e.g., Huggett and Ventura, 1999; Golosov et al., 2016) although some countries do. In fact, in their consideration, they took reduced approach by embedding post-tax public pension benefits in the individual budget of their models. To following this standard, we also assume that $B(y)$ (transfer from government after retirement) is post-tax amount or not subject to income tax.

2.2. Firm and Government

In this economy, there is a representative firm whose technology is linear in labor, following the literature on optimal nonlinear income taxation (e.g., Mirrlees, 1971; Diamond, 1998; Golosov et al., 2016) so that the total output of this economy is

$$Y = (1 - \delta) \int_1^\infty y(\theta) \, dF_\theta = (1 - \delta) \int_1^\infty \theta \, ldF_\theta. \tag{5}$$

At a competitive general equilibrium of this economy, wage rate is equated with the marginal labor product of earning ability, entailing zero economic profit of the firm. The output of the representative firm is used for public and private goods consumption, while only the government can provide public goods. As the final consumption goods, public goods and private goods are perfectly substitutable and the numeraire of this economy. On the other hand, savings are invested outside of this small open economy and the world-wide equilibrium interest rate of r is given to this economy as the rate of return on the savings.

The government of this economy finances public goods provision from collecting income tax on workers with meeting the following fiscal budget constraint:

$$(1 - \delta) \int_1^\infty T(y(\theta)) - \rho y(\theta) dF_\theta = gY = G \tag{6}$$

where $\rho \in (0,1)$ is the public pension contribution rate and $g \in (0,1)$ is an exogenously given portion of the total output earmarked for public goods provision. As progressivity of public pension benefit and labor income tax is the focus of this paper, g is treated as a parameter, although treating g as a choice variable of the government does not change the results of this paper. At the same time, in financing public pension benefits to retirees, the government satisfies the following budget constraint:

$$(1 - \delta) \int_1^\infty \rho \, y(\theta) dF_\theta = \delta \int_1^\infty B\left(y(\theta)\right) dF_\theta. \tag{7}$$

That is, public pension contributions from workers are immediately used to finance public pension benefits for retirees. Thus, the public pension system of this economy is run on a pay-as-you-go basis, as in most of countries that provide public pension such as US, Canada, Austria, and France. Likewise, reflecting the fact that most of public pension systems impose constant linear rate of public pension contribution, the public pension contribution rate of ρ is constant for all levels of pre-tax incomes. To incorporate the fact that public pension benefit formulae of most of public pension systems are nonlinear and relate public pension benefit positively to public pension contribution and thus positively to pre-tax income, we assume that $B(y)$ can be nonlinear and depends positively on y; that is,

$$\frac{dB(y)}{dy} = B'(y) > 0. \tag{8}$$

Although only the inequality of earning ability (marginal labor product) creates income inequality of this economy, earning ability of an individual is not verifiable or observable to the government while his pre-tax income is so. Thus, both public pension benefit and income tax, $B(y)$ and $T(y)$, depend *only* on pre-tax income. On the other hand, the government of this economy does not impose capital income tax.[3]

In particular, the government chooses income tax rate schedule among the following form of nonlinear income tax function

$$T(y) = y - \lambda_L y^{1-\tau_L}. \tag{9}$$

The nonlinear income tax function of (9) has been adopted by various studies such as Feldstein (1969), Persson (1983), Benabou (2000 and 2002), Corneo (2002), Heathcote *et al.* (2017), Serrano-Puente (2020) and the like. Admittedly, in the optimal nonlinear income taxation literature, Mirrlees model (Mirrlees, 1971) is more usual than the nonlinear tax function of (9). However, as mentioned above, Mirrlees model is not suitable for the present analysis, since it is impossible for Mirrlees model to effectively identify optimal marginal public pension benefit rates separately from optimal marginal labor income tax rates or to consistently identify progressivity of public pension benefit (or labor income tax). To see this, notice that the labor distortion of ability-θ individuals of this economy is

$$T'(y) - \frac{1}{(1+r)} \frac{\delta}{(1-\delta)} B'(y) = 1 - \frac{l^\eta}{\theta} c_W \tag{10}$$

which corresponds to the equation (5) of Golosov *et al.* (2006) as well as the equation (13) of Golosov *et al.* (2016). As shown in Golosov *et al.* (2006) and Golosov *et al.* (2016), Mirrlees model identifies only the optimal marginal rate of consumption-leisure substitution, in the right-hand side of (10), to characterize optimal tax/transfer schedule. Thus, Mirrlees model cannot identify optimal marginal public pension benefit rate separately from optimal marginal income tax rate. Because two unknowns with one equation cannot be uniquely

identified, there can be an infinite number of pairs of the optimal rates with no clear criteria to select one. As a consequence, progressivity of public pension benefit cannot be identified under Mirrlees model. To overcome this obstacle, parametric nonlinear functions of public pension benefit and labor income tax are adopted for this paper to effectively identify optimal marginal public pension benefit rate separately from optimal marginal labor income tax rate and to consistently identify optimal degree of progressivity of the entire schedule of labor income tax rates, separately from that of public pension benefit rates. With the nonlinear tax function of (9), progressivity of the entire labor income tax schedule is consistently identified by τ_L.

As post-tax income disposable for private consumption of a worker is $\lambda_L y^{1-\tau_L}$ and $c_W > 0$, $\lambda_L > 0$. Furthermore, if $1 - \tau_L \leq 0$ with $\lambda_L > 0$, working more does not yield more disposable income, causing workers to provide no labor. To induce workers to supply labor and earn positive amount of taxable income,

$$1 - \tau_L > 0 \quad \text{and} \quad \lambda_L > 0. \tag{11}$$

Notably, τ_L indicates progressivity of the entire labor income tax schedule. Specifically, labor income tax is progressive if $\tau_L > 0$ since marginal labor income tax rate increases with pre-tax income if $\dfrac{dT'(y)}{dy} = \lambda_L \tau_L (1 - \tau_L) y^{-\tau_L - 1} > 0$. By the same logic, labor income tax is regressive (i.e., marginal labor income tax rate decreases with pre-tax income) if $\tau_L < 0$. If $\tau_L = 0$, labor income tax rate is linear and the constant marginal labor income tax rate of $1 - \lambda_L$ is imposed equally for all different levels of pre-tax income. Thus, the higher τ_L is, the steeper the slope of the entire labor income tax schedule is. At the same time, average income tax rate is strictly lower (higher) than marginal income tax rate if $\tau_L > 0$ ($\tau_L < 0$).

Note that the nonlinear function of (9) is proven to be very well fitted to the data of the US tax and public pension system by Heathcote, Storesletten and Violante (2017). Because they used "perpetual youth" model where individuals are not retired but live for an infinite number of periods, in their empirical analysis, Heathcote, Storesletten and Violante (2017) used present value of public pension benefits. Thus, their empirical finding on the fitness of (9) suggests that empirically plausible schedules of nonlinear public pension benefit rates can be represented by the same class of functions that (9) belongs to.

In this light, the government selects public pension benefit formula (schedule) among the following form of nonlinear benefit function

$$B(y) = \lambda_P y^{1-\tau_P}. \tag{12}$$

Thus, marginal public pension benefit rate is $B'(y) = \lambda_P (1 - \tau_P) y^{-\tau_P}$ and replacement rate is $\dfrac{B(y)}{y} = \lambda_P y^{-\tau_P}$. As the replacement rate cannot be negative or zero, $\lambda_P > 0$. Moreover, due to (8), $\dfrac{dB(y)}{dy} = \lambda_P (1 - \tau_P) y^{-\tau_P} > 0$ for $\forall y > 0$. Thus,

$$1 - \tau_P > 0 \quad \text{and} \quad \lambda_P > 0. \tag{13}$$

Importantly, τ_P indicates progressivity of the entire public pension benefit schedule. In particular, public pension benefit is progressive if $\tau_P > 0$ since marginal public pension bene-fit rate decreases with pre-tax income if $\dfrac{dB'(y)}{dy} = -\lambda_P \tau_P (1 - \tau_P) y^{-\tau_P - 1} < 0$. By the same logic, public pension benefit is regressive (i. e., marginal public pension benefit rate increases with pre-tax income) if $\tau_P < 0$. If $\tau_P = 0$, public pension benefit rate is linear and the constant marginal public pension benefit rate of λ_P is imposed equally for different levels of pre-tax income. Thus, the higher τ_P is, the steeper the slope of the entire public pension benefit schedule is. At the same time, the replacement rate decreases (increases) with pre-tax income if $\tau_P > 0$ ($\tau_P < 0$) as $\dfrac{d\lambda_P y^{-\tau_P}}{dy} = \lambda_P(-\tau_P)y^{-\tau_P - 1}$.

According to (7) and (12), once τ_P and ρ are decided, λ_P is automatically determined. Thus, for designing the public pension system, the government only needs to choose τ_P and ρ. Likewise, according to (6) and (9), once τ_L and ρ are decided, λ_L is automatically deter-mined. As ρ is public pension contribution rate, for designing the entire labor income tax schedule, the government only needs to choose τ_L. Thus, by choosing τ_P, ρ and τ_L, the gov-ernment completely determines the entire public pension system and labor income tax sched-ule. Moreover, because the government's choice of τ_P, τ_L and ρ is not specific to ability-θ individuals, the incentive constraint does not need to be imposed, even if earning ability of an individual is not observable or verifiable to the government.

As public pension entitlement age from which individuals become eligible to receive public pension benefit is not within the scope of this paper and assumed be fixed in this model. In addition, while voters choose how much to work, they do not decide when to retire. In fact, a great number of empirical studies (e. g., Stock and Wise, 1990; Coile and Gruber, 2007) have shown that most of individuals retire at a given public pension entitlement age. Moreover, the main findings of this paper do not change even if retirement decision is en-dogenous.

The government selects τ_P, τ_L and ρ to maximize the social welfare function, $SW(\{c_W(\theta), c_R(\theta), l(\theta)\}_{\theta=1}^{\infty}; g)$, which is defined as

$$SW = (1 - \beta) \sum_{t=0}^{\infty} \beta^t \int_1^{\infty} \{(1 - \delta)U_W(\theta) + \delta U_R(\theta)\} \, dF_\theta. \tag{14}$$

The value of the social welfare function, which aggregates the utilities of all individu-als of the current and unborn future generations, is determined by the allocation of private consumption and labor supply of individuals. The allocation of private consumption and labor supply is chosen by individuals for themselves, not by the government, and is imple-mented in a decentralized way via competitive market, while it is affected by the govern-ment's choice of τ_P, τ_L and ρ. Thus, an allocation of private consumption and labor supply, which the government intends to induce by its own choice of public pension system and

labor income tax schedule, should be supported as a competitive general equilibrium of this economy. For a given set of the policies of τ_P, τ_L, ρ, λ_P, λ_L and g, a competitive general equilibrium of this economy is defined as the allocation of private consumption and labor supply $\{c_W^*(\theta), c_R^*(\theta), l^*(\theta)\}_{\theta=1}^{\infty}$ that satisfies the following three conditions. (i) With the government policies and price given, the allocation of private consumption and labor supply maximizes the utility of each individual meeting their own budget constraint; (ii) the profit of the representative firm is maximized and factor market is cleared by equating wage rate with marginal labor product (earning ability); (iii) the two government budget constraints of (6) and (7) are met. According to Walras' law, once this economy reaches its competitive equilibrium by meeting these three conditions of (i), (ii) and (iii), goods market of this economy is automatically cleared.

2.3. Competitive-Equilibrium Allocation in Terms of Policy Variable

For characterizing competitive-equilibrium allocation, pre-retirement private consumption as a portion of post-tax labor income is stated as $s_W \lambda_L y^{1-\tau_L} = c_W$. Likewise, public pension benefit as a portion of post-retirement private consumption is stated as $s_R c_R = \lambda_P y^{1-\tau_P}$. Then, for any given θ, embedding the budget constraints of (3) and (4), which bind at a competitive general equilibrium, the utility of an ability-θ worker is restated as

$$log(s_W \lambda_L (\theta l)^{1-\tau_L}) - \frac{l^{1+\eta}}{1+\eta} + \frac{\beta\delta}{1-\delta} log(\frac{\lambda_P(\theta l)^{1-\tau_P}}{s_R}) + \chi(1 + \frac{\beta\delta}{1-\delta}) log(G) \quad (15)$$

and the utility of an ability-θ retiree as

$$log(\frac{\lambda_P(\theta l)^{1-\tau_P}}{s_R}) + \chi \, log(G). \quad (16)$$

From maximizing (15), the optimal level of labor supply of an ability-θ worker is defined as follows: for any given θ,

$$l^*(\theta) = \{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{1}{1+\eta}}. \quad (17)$$

As shown in (17), an increase in the degree of income tax progressivity (an increase in τ_L) reduces workers' labor supply by $\frac{1}{1+\eta}\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{1}{1+\eta}-1}$ with decreasing the current (pre-retirement) return on labor supply. Moreover, a worker supplies his labor not only to earn the current income for pre-retirement consumption (c_W) but also to increase public pension benefit for post-retirement consumption (c_R). In this light, an increase in the degree of public pension benefit progressivity (an increase in τ_P) reduces workers' labor supply by $(\frac{\beta\delta}{1-\delta})\frac{1}{1+\eta}\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{1}{1+\eta}-1}$, decreasing the future (post-retirement) return on labor supply. Notice that an increase in the degree of public pension benefit progres-

sivity reduces workers' labor supply by smaller margin than the same-size increase in the degree of income tax progressivity does. At the same time, maximizing the utility of an ability-θ worker also yields the following Euler equation of consumption smoothing: for any given θ,

$$\frac{1}{c_W^*(\theta)} = \frac{\beta\delta}{1-\delta}(1+r)\frac{1}{c_R^*(\theta)} \tag{18}$$

which is equivalent to $\frac{1}{s_R^*(\theta)} = \frac{\beta\delta}{1-\delta}(1+r)\frac{\lambda_L(\theta l^*(\theta))^{\tau_P-\tau_L}}{\lambda_P}s_W^*(\theta)$. Since $(1-s_R)c_R = (1+r)k = (1+r)(1-s_W)\lambda_L y^{1-\tau_L}$ due to the budget constraints of (3) and (4), (18) implies that for any given θ, $s_W^*(\theta) = \dfrac{(1+r)\lambda_L(\theta l^*(\theta))^{1-\tau_L} + \lambda_P(\theta l^*(\theta))^{1-\tau_P}}{(1+\dfrac{\beta\delta}{1-\delta})(1+r)\lambda_L(\theta l^*(\theta))^{1-\tau_L}}$. As $c_W^*(\theta) = s_W^*(\theta)\lambda_L(\theta l^*(\theta))^{1-\tau_L}$,

$$c_W^*(\theta) = \frac{\lambda_L(\theta l^*(\theta))^{1-\tau_L} + \dfrac{\lambda_P(\theta l^*(\theta))^{1-\tau_P}}{(1+r)}}{(1+\dfrac{\beta\delta}{1-\delta})} \tag{19}$$

which in turn implies from (18) that $c_R^*(\theta) = \dfrac{\beta\delta}{(1-\delta)}\dfrac{(1+r)\lambda_L(\theta l^*(\theta))^{1-\tau_L} + \lambda_P(\theta l^*(\theta))^{1-\tau_P}}{(1+\dfrac{\beta\delta}{1-\delta})}$.

Such allocation of $\{c_W^*(\theta), c_R^*(\theta), l^*(\theta)\}_{\theta=1}^{\infty}$ that meets (17), (18) and (19) maximizes the utility of all individuals meeting their own budget constraint as well as clears factor market equating wage rate with marginal labor product (earning ability).

From (5) and (17), the competitive-equilibrium total output of this economy is

$$Y^* = (1-\delta)\int_1^{\infty} \theta\, l^*(\theta)dF_\theta = (1-\delta)\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{1}{1+\eta}}\frac{\alpha}{\alpha-1} \tag{20}$$

from which we can define $gY^* = G^*$ for the last terms of (15) and (16). Then, with $\{c_W^*(\theta), c_R^*(\theta), l^*(\theta)\}_{\theta=1}^{\infty}$ and G^*, defined from (17), (18), (19) and (20), the social welfare function of (14) is restated in terms of the government's choice variables of τ_P, τ_L, λ_P and λ_L satisfying the conditions (i) and (ii) for being implemented as a competitive general equilibrium. Having embedded the first two conditions for competitive-equilibrium allocation into the social welfare function, the last condition (iii) of satisfying the two budget constraints of the government, (6) and (7), also can be incorporated into the social welfare function. To this end, according to (6) and (7), λ_P and λ_L are restated in terms of τ_P, τ_L and ρ. Based on (7), (17) and (20),

$$\lambda_P = \frac{\rho(1-\delta)}{\delta}\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{\tau_P}{1+\eta}}\frac{(\alpha+\tau_P-1)}{\alpha-1}. \tag{21}$$

And, based on (6), (17), and (20),

$$\lambda_L = [1 - \rho - \frac{g}{(1-\delta)}]\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{\tau_L}{1+\eta}}\frac{(\alpha+\tau_L-1)}{\alpha-1}. \qquad (22)$$

Now, using (17), (18) and (19), the indirect utility functions of workers and retirees are written in terms of τ_P, τ_L, λ_P and λ_L, by which the conditions of (i) and (ii) are satisfied. Then, using (20), (21) and (22), the indirect utility functions are restated only in terms of τ_P, τ_L and ρ, by which the condition (iii) is also met. Lastly, aggregating the restated indirect utility functions of workers and retirees with the population weight yields the restated social welfare value function of $SV(\tau_P, \tau_L, \rho; g)$ that satisfies all the three conditions for being supported as a competitive equilibrium.

$$SV = \{(1-\delta)(1+\frac{\beta\delta}{1-\delta}) + \delta\}[log(\frac{\alpha+\tau_P-1}{\alpha-1}) + log(\frac{\alpha+\tau_L-1}{\alpha-1}) +$$

$$\frac{2-\tau_P-\tau_L}{\alpha} + \frac{(1+\chi)}{1+\eta}log((1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)) + log(1-\rho-\frac{g}{1-\delta}) +$$

$$log(\rho)] - (1-\delta)\frac{1}{1+\eta}\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\} + \{(1-\delta)(1+\frac{\beta\delta}{1-\delta}) + \delta\} \quad (23)$$

$$[\chi log(g) + log((\frac{\alpha(1-\delta)}{\alpha-1})^{\chi}(1+\frac{\beta\delta}{1-\delta})^{-1})] + log(\frac{1-\delta}{\delta}) +$$

$$\beta\delta log(\beta) + \delta log(\frac{\beta\delta}{1-\delta}) + (\delta-1)log(1+r)$$

From (23) notice the separation between g and the government's choice variables of τ_P and τ_L which implies that the optimal degree of progressivity of public pension benefit and labor income tax is not affected by whether g is exogenously given or endogenously chosen. More importantly, the government can identify optimal public pension system and labor income tax schedule that are implementable as a competitive general equilibrium just by maximizing $SV(\tau_P, \tau_L, \rho; g)$ (unconstrained maximization problem).

3. Characterization of Optimal Progressivity of Public Pension Benefit and Labor Income Tax

3.1. Optimal Schedule and Progressivity of Public Pension and Labor Income Tax

Having described our economy, we now characterize the optimal public pension system and labor income tax schedule (τ_P^*, τ_L^*, ρ^*, λ_P^* and λ_L^*) that maximize the social welfare function. As elaborated above, the optimal public pension system and income tax schedule can be identified by maximizing $SV(\tau_P, \tau_L, \rho; g)$, whose consequent allocation of individuals' private consumption and labor supply is implemented in decentralized way via competitive market. By obtaining the optimal values of τ_P^*, τ_L^*, ρ^*, λ_P^* and λ_L^* for ability-θ individuals, optimal marginal public pension benefit rate and optimal marginal labor income tax rate for each

individual are uniquely identified according to (12) and (9) respectively. Above all, from the social-welfare maximization, optimal degree of progressivity of public pension benefit and labor income tax (τ_P^* and τ_L^*), which is of our focus, is derived.

Proposition 1. The optimal public pension schedule (τ_P^*, ρ^* and λ_P^*) is defined by

$$(1+\beta\delta)\frac{(1-\tau_P^*)}{\alpha(\alpha+\tau_P^*-1)} = \frac{\beta\delta}{(1-\delta)}[\frac{(1+\beta\delta)\frac{(1+\chi)}{(1+\eta)}}{\{(1-\tau_L^*)+\frac{\beta\delta}{1-\delta}(1-\tau_P^*)\}} - \frac{(1-\delta)}{(1+\eta)}], \quad (24)$$

$$\rho^* = \frac{1}{2}(1-\frac{g}{1-\delta}), \quad (25)$$

$$\lambda_P^* = \frac{\rho^*(1-\delta)}{\delta}\{(1-\tau_L^*)+\frac{\beta\delta}{1-\delta}(1-\tau_P^*)\}^{\frac{\tau_P^*}{1+\eta}}\frac{(\alpha+\tau_P^*-1)}{\alpha-1}, \quad (26)$$

while the optimal labor income tax schedule (τ_L^* and λ_L^*) is defined by

$$(1+\beta\delta)\frac{(1-\tau_L^*)}{\alpha(\alpha+\tau_L^*-1)} = \frac{(1+\beta\delta)\frac{(1+\chi)}{(1+\eta)}}{\{(1-\tau_L^*)+\frac{\beta\delta}{1-\delta}(1-\tau_P^*)\}} - \frac{(1-\delta)}{(1+\eta)}, \quad (27)$$

$$\lambda_L^* = [1-\rho^*-\frac{g}{(1-\delta)}]\{(1-\tau_L^*)+\frac{\beta\delta}{1-\delta}(1-\tau_P^*)\}^{\frac{\tau_L^*}{1+\eta}}\frac{(\alpha+\tau_L^*-1)}{\alpha-1}. \quad (28)$$

For proof, see Appendix A1.

Notably, *Proposition 1* demonstrates the underlying forces that shape the optimal progressivity of public pension benefit and labor income tax. To see this, first notice from (14) or (23) that $1+\beta\delta = \{(1-\delta)(1+\frac{\beta\delta}{1-\delta})+\delta\}$ is social marginal utility from a change in private consumption of all individuals (workers and retirees) for their remaining lifetime. Thus, the left-hand side of (24) is the social marginal benefit from an increase in the degree of public pension benefit progressivity (i.e., an increase in τ_P) that reduces consumption inequality by reducing post-retirement income inequality among all individuals. Because the social welfare function is concave in consumption, a decrease in the consumption gap between the rich and the poor (between high-ability individuals and low-ability individuals) brought by an increase in the degree of public pension benefit progressivity entails a social-welfare gain. As shown in (18), individuals smooth consumption; hence, an increase in the degree of progressivity of public pension benefit leads to a decrease in the inequality of both pre-retirement and post-retirement consumption, although it reduces only the inequality of post-retirement income. The left-hand side of (24) also reveals that the social marginal benefit of this equity enhancement brought by an increase in τ_P depends positively on the pre-government inequality, since the Gini index of unequally endowed earning abilities is $\frac{1}{2\alpha-1}$.

While the welfare improvement of alleviating the inequality of income and consumption by an increase in τ_P exerts upward force for choosing more progressive public pension benefit, labor-supply disincentive from the increase in τ_P puts downward pressure on it. In particular, an increase in the degree of public pension benefit progressivity decreases workers' labor supply by reducing the future return (post-retirement return) on their labor supply. The first term of the right-hand side of (24) represents the social marginal cost that an increase in the degree of public pension benefit progressivity reduces the total output available for private and public goods consumption by decreasing workers' labor supplies. At the same time, as an increase in τ_P decreases workers' labor supplies, it also reduces workers' disutility of labor supply, cancelling part of the social cost of the efficiency loss on the total output, as indicated by the second term of the right-hand side of (24). Taking the balance between the opposite forces of equity enhancement and net efficiency loss, the formula of (24) shows that the optimal degree of public pension benefit progressivity is selected to equate the social marginal benefit of equity improvement from an increase in τ_P (the left-hand side) with its net social marginal cost of efficiency loss (the right-hand side).[4]

On the other hand, the formula of (27) shows that the forces shaping the optimal progressivity of labor income tax are similar to those shaping the optimal progressivity of public pension benefit. As shown in the left-hand side of (27), an increase in the degree of income tax progressivity yields the social marginal benefit by alleviating pre-retirement income inequality. This upward force for more progressive income tax is countered by the downward force of the labor-supply reduction. By decreasing the current (pre-retirement) return on labor supply, an increase in τ_L reduces workers' labor supplies, which decreases the total output for private and public goods consumption and decreases workers' disutility of labor supply, as shown in the right-hand side of (27). Taking together, (27) shows that the optimal degree of labor income tax progressivity is chosen to equate the social marginal benefit from an increase in τ_L (the left-hand side) with its net social marginal cost (the right-hand side). Although the trade-off between equity and efficiency shapes both of the optimal degree of public pension benefit progressivity and the optimal degree of labor income tax progressivity, the exact way how it shapes the respective optimal progressivity is different. It is the future (post-retirement) return on labor supply through which public pension benefit progressivity incurs efficiency loss for alleviating inequality, while it is the current (pre-retirement) return on labor supply through which labor income tax progressivity does.

As shown in *Proposition 1*, both τ_P and τ_L alike lead to equity enhancement by alleviating inequality among individuals. To delineate redistributive effect of τ_P and τ_L, we can compare Gini indexes of pre-government and post-government income. On the one hand, for any given τ_P and τ_L, the Gini index of pre-government income $y = \theta l^*$ is $\dfrac{1}{(2\alpha - 1)}$, which is equal to the Gini index of endowed earning abilities. On the other hand, for any given τ_P and τ_L, lifetime income after tax and public pension benefit transfer (post-government income) of an individual is $\lambda_L(\theta l^*)^{1-\tau_L} + \dfrac{1}{1+r}\lambda_P(\theta l^*)^{1-\tau_P}$ which is equated with his lifetime con-

sumption $c_W^*(\theta) + \dfrac{1}{1+r} c_R^*(\theta)$, as his budget constraint binds at a competitive equilibrium.

To straightforwardly show the redistributive effect of τ_P and τ_L, respectively, the Gini index

of $\lambda_L(\theta l^*)^{1-\tau_L}$ and that of $\dfrac{1}{1+r}\lambda_P(\theta l^*)^{1-\tau_P}$ are calculated according to Atkinson (1970).

For details of the calculation, refer to Appendix A2. Obviously, a decrease in the inequality of post-tax labor income $\lambda_L(\theta l^*)^{1-\tau_L}$ or a decrease in the inequality of the present value of

public pension benefit income $\dfrac{1}{1+r}\lambda_P(\theta l^*)^{1-\tau_P}$ always reduces the inequality of post-government income. Firstly, for any given τ_P and τ_L, the Gini index of post-tax labor income of $\lambda_L(\theta l^*)^{1-\tau_L}$ is

$$\frac{(1-\tau_L)}{2\alpha + \tau_L - 1} \tag{29}$$

which can be reduced by an increase in the degree of labor income tax progressivity since $\dfrac{d}{d\tau_L}\{\dfrac{(1-\tau_L)}{2\alpha + \tau_L - 1}\} = \dfrac{-2\alpha}{(2\alpha + \tau_L - 1)^2} < 0$. Thus, labor income tax progressivity exerts positive redistributive effect by reducing the inequality of post-tax labor income of individuals of different earning abilities. Secondly, for any given τ_P and τ_L, the Gini index of the present

value of public pension benefit income of $\dfrac{1}{1+r}\lambda_P(\theta l^*)^{1-\tau_P}$ (the remaining part of the total post-government income) is

$$\frac{(1-\tau_P)}{2\alpha + \tau_P - 1} \tag{30}$$

which can be reduced by an increase in the degree of public pension benefit progressivity as $\dfrac{d}{d\tau_P}\{\dfrac{(1-\tau_P)}{2\alpha + \tau_P - 1}\} = \dfrac{-2\alpha}{(2\alpha + \tau_P - 1)^2} < 0$. Hence, the redistributive effect of public pension benefit progressivity is also positive and is achieved by reducing the inequality of public pension benefit income of individuals of different earning abilities. Basically, both public pension benefit progressivity and labor income tax progressivity alleviate the lifetime income inequality within generation as well as the current income inequality within and across generations (workers and retirees) at any given time.

Most of all, putting (29) and (30) together shows that the redistributive effect of public pension benefit progressivity is complementary to that of labor income tax progressivity. An increase in the degree of public pension benefit progressivity results in lower level of lifetime and current income inequality at a higher degree of income tax progressivity. Likewise, an increase in the degree of labor income tax progressivity ends up with lower level of lifetime and current income inequality at a higher degree of public pension benefit progressivity. In

addition, either τ_P or τ_L cannot perfectly substitute each other. For instance, no feasible value of τ_P can replace $\tau_L = 1$ to achieve no inequality of post-tax labor income.

Note that pay-as-you-go financing in itself does not effectively reduce inequality since by paying public pension contribution workers are entitled to receive public pension benefit after they retire, although their contribution is immediately used for the current retirees. For instance, with linear rate of public pension contribution (ρ), if public pension benefit is designed to assign strictly higher replacement rates to high-ability individuals than to low-ability individuals (i.e., if $\tau_P < 0$), pay-as-you-go financing may end up with exacerbating income inequality rather than alleviating it. Thus, the value of τ_P (progressivity of public pension benefit) is crucial for whether the pay-as-you-go public pension system alleviates income inequality. While an increase in the degree of labor income tax progressivity directly reduces workers' income inequality, it indirectly reduces retirees' income inequality by making the amount of public pension contribution more equal. Similarly, while an increase in the degree of public pension benefit progressivity directly reduces retirees' income inequality, it indirectly reduces workers' income inequality by decreasing the post-retirement return on high-ability workers' labor supply more than that on low-ability workers'. As such, the inter-generational income inequality between rich workers and poor retirees or between poor workers and rich retirees is alleviated by both of τ_P and τ_L together.

3.2. Optimal Relation of Public Pension Benefit and Income Tax Progressivity

As shown in *Proposition 1*, the exact values of τ_P^* and τ_L^* are mutually depend on each other. Nevertheless, how progressivity of public pension benefit should be related to progressivity of labor income tax is not immediately clear from *Proposition 1* by itself. Because the underlying forces that shape each degree of progressivity are almost the same except for whether the current or future return on labor supply is affected by an increase in the degree of progressivity, one may well guess that τ_P^* is related to τ_L^* in the same way of present-value discounting by which future return is converted to current return at a constant rate like $\dfrac{1}{(1+r)}$. Based on this conjecture, define h as $h\tau_L = \tau_P$ so that the relation of public pension benefit progressivity and labor income tax progressivity is indicated by h. To find whether this conjecture is true or not and to clarify the optimal relation of public pension benefit progressivity and labor income tax progressivity, the optimal ratio of h^* (the ratio of τ_L^* to τ_P^*) is derived from *Proposition 1*.

Proposition 2. The optimal ratio of the degree of public pension benefit progressivity to that of labor income tax progressivity is

$$h^* = \frac{(\alpha - 1)\dfrac{\beta\delta}{(1-\delta)} - 1 + (\dfrac{\beta\delta}{1-\delta} - 1)\tau_P^*}{\dfrac{\beta\delta}{1-\delta} + (\alpha - 1) - (1 - \dfrac{\beta\delta}{1-\delta})(\alpha - 1)\dfrac{1}{\tau_P^*}} = \frac{\tau_P^*}{\tau_L^*}. \tag{31}$$

Moreover, the defining property of the optimal ratio is

$$
\frac{\frac{\beta\delta}{1-\delta}}{1} = \frac{(1+\beta\delta)[\dfrac{1-\tau_P^*}{\alpha(\alpha+\tau_P^*-1)}]}{(1+\beta\delta)[\dfrac{1-\dfrac{\tau_P^*}{h^*}}{\alpha(\alpha+\dfrac{\tau_P^*}{h^*}-1)}]} = \frac{(1+\beta\delta)[\dfrac{1-\tau_P^*}{\alpha(\alpha+\tau_P^*-1)}]}{(1+\beta\delta)[\dfrac{1-\tau_L^*}{\alpha(\alpha+\tau_L^*-1)}]}.
\tag{32}
$$

For proof, see Appendix A3.

Rebutting our conjecture, (31) of *Proposition 2* shows that the optimal ratio of public pension benefit progressivity to labor income tax progressivity is not constant as present-value-discounting factor but varies depending on the chosen degree of public pension benefit progressivity, the pre-government inequality, and the population share of retirees, in a complicated way. Furthermore, while the ratio of public pension benefit progressivity to labor income tax progressivity is applied uniformly for different individuals of different earning abilities, the ratio of marginal public pension benefit rate to marginal labor income tax rate,

which is $\dfrac{\lambda_P^*(1-\tau_P^*)(y^*)^{-\tau_P^*}}{1-\lambda_L^*(1-\tau_L^*)(y^*)^{-\tau_L^*}}$, varies across different individuals of different earning abil-

ities. This may make understanding of the optimal h^* even more complicated.

After all, (32) of *Proposition 2* reveals that the optimal ratio of public pension benefit progressivity to labor income tax progressivity is selected to entail *no* distortion on inter-temporal allocation of individuals. To see this, in light of *Proposition 1*, the numerator of the right-hand side of (32) is the social marginal benefit of reducing the post-retirement income inequality by an increase in τ_P, while the denominator of the right-hand side of (32) is the

social marginal benefit of reducing the pre-retirement income inequality by an increase in τ_L. Notice that $\dfrac{\beta\delta}{1-\delta}$ is the ratio of the current marginal utility to the future marginal utility of

workers who are only *part* of the entire population of this economy, whereas the ratio of so-

cial marginal utility of the current period to that of the next period is β, not $\dfrac{\beta\delta}{1-\delta}$, as shown in

the social welfare function of (14). The ratio of the current marginal utility to the future marginal utility of workers is crucial for workers' inter-temporal allocation of their own choice of their savings (pre-retirement and post-retirement consumption) as well as their own assessment of the current and future return on their labor supplies in responding to a given degree of progressivity of public pension benefit and labor income tax. In the terminal period of their life, retirees do not need to choose inter-temporal allocation, in contrast to workers. If the government chooses the ratio of the social marginal benefit of post-retirement income pro-

gressivity to that of pre-retirement income progressivity to be different from $\dfrac{\beta\delta}{1-\delta}$, then the

inter-temporal allocation of workers is distorted even with no capital income tax. Therefore,

the ratio of public pension benefit progressivity to labor income tax progressivity is selected to equate the post-government rate of substitution between current marginal utility and future marginal utility of workers with the pre-government rate of substitution between their current and future marginal utility, for entailing *no* distortion on the inter-temporal allocation. While the optimal income tax and public pension distort the intra-temporal allocation of workers by reducing their labor supply for welfare-improving equity enhancement, the optimal relation of public pension benefit progressivity and income tax progressivity is arranged not to distort the inter-temporal allocation, to minimize the overall efficiency loss.

Although *Proposition 2* delineates the optimal relation between public pension benefit progressivity and labor income tax progressivity, it is not yet evident whether the optimal public pension benefit is more progressive than the optimal labor income tax, which in itself is an important policy issue. In fact, based on *Proposition 1*, at the current level of generality, we can show that the optimal ratio of public pension benefit progressivity to labor income tax progressivity (h^*) is strictly higher than one.

Proposition 3. The optimal public pension benefit is more progressive than the optimal labor income tax. That is, $\tau_P^* > \tau_L^*$.

For proof, see Appendix A4.

As shown by (17), an increase in the degree of public pension benefit progressivity reduces workers' labor supply by smaller margin than the same-size increase in the degree of labor income tax progressivity does, while each increase in the degree of progressivity alleviates inequality. Hence, to achieve equity improvement, efficiency loss of labor-supply reduction from an increase in the degree of public pension benefit progressivity is relatively smaller than that from the same-size increase in the degree of labor income tax progressivity, which leads the optimal public pension benefit to be more progressive than the optimal labor income tax. Thus, at the social optimum, the social-welfare maximizing government alleviates the inter-generational income inequality between rich workers and poor retirees or between poor workers and rich retirees by reducing income gap between rich retirees and poor retirees by greater extent than income gap between rich workers and poor workers.

As one of the policy-relevant results of this paper, *Proposition 3* holds independent of the elderly population share or the pre-government inequality, as shown in its proof. Moreover, *Proposition 2 and 3* imply that the optimal marginal public pension benefit rate for ability-θ individuals is not converted to the optimal marginal labor income tax rate for them at a constant rate that is uniformly applied to all individuals. Therefore, the optimal public pension benefit schedule cannot be regarded as trivial extensions of the vast literature on optimal labor income tax schedule.

So far, from identifying the optimal public pension system and optimal labor income tax schedule, the optimal progressivity of public pension benefit and that of labor income tax is characterized. This characterization paves the way for analyzing how the pre-government inequality and population aging, respectively, affect the optimal degree of pro-

gressivity of public pension benefit and labor income tax, which is presented in the next section.

4. Effects of Pre-Government Inequality and Population Aging on the Optimal Progressivity of Public Pension Benefit and Labor Income Tax

In the previous section, the optimal schedule of pay-as-you-go public pension benefit and labor income tax is derived for a *given* level of the pre-government inequality and of the elderly population share. Recently, we face the problems of rapidly aging population (e. g., United Nations, 2017) and rises in the pre-tax income inequality (e. g., OECD, 2011). Hence, it is worthwhile to examine how the pre-government inequality and population aging, respectively, make a difference in the optimal degree of progressivity of public pension benefit and labor income tax.

To this purpose, recall that for any given τ_P and τ_L, the Gini index of pre-government income is $\dfrac{1}{(2\alpha - 1)}$ which is equal to the Gini index of unequally endowed earning abilities. Thus, by introducing a variation in the value of the parameter of α, we can analyze the effect of the pre-government inequality on the optimal progressivity of public pension benefit and labor income tax, respectively, based on *Proposition 1*.

Proposition 4. The pre-government inequality positively affects the optimal degree of progressivity of public pension benefit as well as the optimal degree of progressivity of labor income tax.

For proof, see Appendix A5.

Basically, as appears in the left-hand side of (24), an increase in the level of the pre-government inequality (i. e., a decrease in α) raises the social marginal benefit from an increase in the degree of public pension benefit progressivity, causing the optimal degree of public pension benefit progressivity (τ_P^*) to increase. Likewise, as shown in the left-hand side of (27), an increase in the pre-government inequality raises the social marginal benefit from an increase in the degree of labor income tax progressivity, leading to an increase in the optimal degree of labor income tax progressivity (τ_L^*). The higher is the pre-government inequality, the larger gap is in the pre-government marginal utility between the rich and the poor (i. e., between high-ability and low-ability individuals), causing a given degree of progressivity of public pension benefit and labor income tax to yield more welfare gain.

Moreover, even after an increase in the level of the pre-government inequality, the optimal public pension benefit is more progressive than the optimal labor income tax, because the proof of *Proposition 3* is valid independent of the value of α. On the other hand, the effect of the pre-government inequality on the optimal ratio of public pension benefit progressivity to

labor income tax progressivity (h^*) is not certainly determinable.[5] As appears in the numerator and denominator of (32) of *Proposition 2*, an increase in the level of the pre-government inequality (a decrease in α) simultaneously raises the social marginal benefit of reducing post-retirement income inequality and the social marginal benefit of reducing pre-retirement income inequality; hence, it is theoretically ambiguous whether an increase in the level of the pre-government inequality can increase the ratio of τ_P^* to τ_L^*.

In addition, although our model does not include unemployment episode, introducing unemployment can be translated into a further variation in pre-tax incomes that means an increase in the pre-government inequality. Therefore, based on *Proposition 2*, we can expect that introducing unemployment episode will also positive affect the optimal progressivity of public pension and labor income tax.

As population aging is represented by a rise in the population share of the elderly, by introducing an increase in the value of δ, we can examine the effect of population aging on the optimal progressivity of public pension benefit and labor income tax.

Proposition 5. The effect of population aging on the optimal degree of progressivity of public pension benefit and on the optimal degree of labor income tax progressivity is ambiguous.

For proof, see Appendix A6.

As the left-hand sides of (24) and (27) of *Proposition 1* increase with δ, population aging makes an increase in the degree of progressivity of public pension benefit and labor income tax, respectively, bring larger social marginal benefit, exerting upward pressure on the optimal progressivity of public pension benefit and labor income tax. As population aging raises the elderly population share, an increase in the degree of public pension benefit progressivity directly reduces the post-retirement income inequality of increased number of retirees, while an increase in the degree of labor income tax progressivity indirectly does so. At the same time, however, population aging causes an increase in the degree of progressivity of public pension benefit and labor income tax, respectively, to incur more efficiency loss, as the right-hand sides of (24) and (27) increase with δ. Apparently, population aging reduces the population share of workers who produce the total output for all individuals of this economy, as shown in (20). By shrinking the labor force, population aging necessitates giving larger labor-supply incentives for the social-welfare maximization, exerting downward pressure on the optimal degree of progressivity of public pension benefit and labor income tax. Which one of these two opposite pressures dominates is not a priori determinable at the current level of generality; hence, it is ambiguous whether population aging increases or decreases the optimal degree of progressivity of public pension benefit and labor income tax.

Likewise, the effect of population aging on the optimal ratio of public pension benefit progressivity to labor income tax progressivity (h^*) is not certainly determinable, either. However, because the proof of *Proposition 3* is valid independent of the value of δ, even after an increase in the population share of retirees, the optimal public pension benefit is unambiguously more progressive than the optimal labor income tax.

In fact, all the theoretical findings of this paper also hold when earning ability is distributed according to Lognormal distribution that is frequently adopted in the literature on income distribution and has thinner upper tails than Pareto distribution. As a robustness chock, the proofs of *Proposition 1, 2, 3, 4 and 5* with Lognormal distribution of earning ability are presented in Appendix B1 and B2.

5. Concluding Remarks

In sum, this paper characterizes optimal progressivity of public pension benefit and labor income tax in an overlapping-generations model with pay-as-you-go public pension. Trading off the efficiency loss of reducing labor supply for the equity enhancement of reducing intra- and inter-generational inequality shapes the optimal progressivity of public pension benefit and labor income tax. The optimal ratio of the degree of progressivity of public pension benefit to that of labor income tax is not constant as the present-value-discounting factor but varies depending on the adopted degree of progressivity of public pension benefit, the pre-government inequality, and the elderly population share. The optimal ratio is selected for entailing no inter-temporal allocation distortion. Importantly, this paper shows that the optimal public pension benefit is more progressive than the optimal labor income tax. Moreover, this paper also theoretically proves that while an increase in the level of the pre-government inequality makes the optimal public pension benefit and labor income tax more progressive, the effect of population aging on the optimal progressivity of public pension benefit and labor income tax is ambiguous.

The above theoretical findings of this article can have various policy implications for numerous economies. This paper immediately suggests that policy debates on income distribution should consider progressivity of public pension benefit and labor income tax *together* although current policy debates actually ignore the inter-dependency of the two. Moreover, for minimizing efficiency loss in achieving equity, this paper recommends against the policy design of linear public pension benefit rates and nonlinear (progressive) labor income tax, because public pension benefit schedule should be more progressive than labor income tax schedule. For example, unless labor income tax is linear, linear public pension benefit rates of some notional defined contribution public pension system, as described by Fehr *et al*. (2013), is sup-optimal. In addition, for a country policy making to address population aging, the model of this paper can be utilized for calculating how population aging changes optimal degree of public pension and income tax. For such calculation of numerical simulation to be useful, careful estimation of the proper parameter values with micro- and macro- data of the country is necessary, which is related to future studies as it is out of the scope of this paper.

Appendix A

A1. Proof of Proposition 1

To begin, the social welfare value function is concave in τ_P, ρ and τ_L, because $\dfrac{d^2 SV}{d\tau_P{}^2} < 0$

for any feasible value of τ_P according to (13), $\dfrac{d^2 SV}{d\rho^2} < 0$ for $\forall \rho \in (0,1)$, and $\dfrac{d^2 SV}{d\tau_L{}^2} < 0$ for

any feasible value of τ_L according to (11). Hence, for identifying the optimal values of τ_P^*, ρ^*, λ_P^*, τ_L^* and λ_L^* that define optimal public pension system and labor income tax schedule, the following three first-order conditions are sufficient. First, the first-order condition for τ_P^* is

$$\frac{dSV}{d\tau_P} = (1 + \beta\delta)\left[\frac{1}{\alpha + \tau_P^* - 1} - \frac{1}{\alpha}\frac{(1 + \chi)}{(1 + \eta)}\frac{\frac{\beta\delta}{(1-\delta)}}{\{(1 - \tau_L^*) + \frac{\beta\delta}{1-\delta}(1 - \tau_P^*)\}}\right] + \frac{\beta\delta}{1 + \eta} = 0 \quad (A1)$$

which is equivalent to (24). Second, the first-order condition for ρ^* is

$$\frac{dSV}{d\rho^*} = (1 + \beta\delta)\left\{\frac{-1}{1 - \rho^* - \frac{g}{1-\delta}} + \frac{1}{\rho^*}\right\} = 0 \quad (A2)$$

which entails (25). Third, the first-order condition for τ_L^* is

$$\frac{dSV}{d\tau_L^*} = (1 + \beta\delta)\left[\frac{1}{\alpha + \tau_L^* - 1} - \frac{1}{\alpha}\frac{(1 + \chi)}{(1 + \eta)}\frac{1}{\{(1 - \tau_L^*) + \frac{\beta\delta}{1-\delta}(1 - \tau_P^*)\}}\right] + \frac{(1 - \delta)}{1 + \eta} = 0 \quad (A3)$$

which is equivalent to (27). Once the optimal values of τ_P^*, ρ^* and τ_L^* are identified from (A1), (A2) and (A3), the optimal values of λ_P^* and λ_L^* are definitely identified according to (21) and (22), which completes the proof.

A2. Calculation of the Gini Index for Post-tax Labor Income and Present-discounted Public Pension Benefit Income

According to Atkinson (1970), recall that the formula of Gini index of income x is

$$\frac{2}{\mu_x}\left[\int_0^\infty x\, F(x)f(x)dx - \frac{\mu_x}{2}\right] \quad (A4)$$

where $\mu_x = E[x]$ is the average income; $F(x)$ is the cumulative distribution function (CDF) of income x; $f(x)$ is the probability density function (PDF) of income x. In fact, this formula is also used for obtaining the Gini index of pre-government income θl^*. To apply the formula of (A4) for post-tax labor income of $\lambda_L(\theta l^*)^{1-\tau_L}$, due to (17), we first obtain $E\left[\lambda_L(\theta l^*)^{1-\tau_L}\right]$ as follows:

$$E[\lambda_L(\theta l^*)^{1-\tau_L}] = \lambda_L\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{1-\tau_L}{1+\eta}} \frac{\alpha}{\alpha + \tau_L - 1}. \tag{A5}$$

Since the PDF of post-tax labor income is $\frac{\alpha}{\theta^{\alpha+1}}$ and the CDF is $1 - \frac{1}{\theta^\alpha}$,

$$\int_1^\infty \lambda_L(\theta l^*)^{1-\tau_L} \left\{\frac{\alpha}{\theta^{\alpha+1}} - \frac{\alpha}{\theta^{2\alpha+1}}\right\} d\theta =$$

$$\lambda_L\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{1-\tau_L}{1+\eta}} \int_1^\infty (\theta)^{1-\tau_L} \left\{\frac{\alpha}{\theta^{\alpha+1}} - \frac{\alpha}{\theta^{2\alpha+1}}\right\} d\theta. \tag{A6}$$

When plugging (A5) and (A6) into the formula of (A4), $\lambda_L\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{1-\tau_L}{1+\eta}}$

is cancelled out so that the Gini index of post-tax labor income of $\lambda_L(\theta l^*)^{1-\tau_L}$ is equal to (29). By the same logic, the average present value of public pension benefit income is

$$E[\frac{1}{1+r}\lambda_P(\theta l^*)^{1-\tau_P}] = \frac{1}{1+r}\lambda_P\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{1-\tau_P}{1+\eta}} \frac{\alpha}{\alpha + \tau_P - 1}. \tag{A7}$$

In using the formula of (A4) to get the Gini index of the present value of public pension benefit income of $\frac{1}{1+r}\lambda_P(\theta l^*)^{1-\tau_P}$ as (30), $\frac{1}{1+r}\lambda_P\{(1-\tau_L) + \frac{\beta\delta}{1-\delta}(1-\tau_P)\}^{\frac{1-\tau_P}{1+\eta}}$ is cancelled out. As shown in Section 3, neither λ_P nor λ_L affects income inequality directly, while τ_L and τ_P do. Above all, the Gini indexes of (29) and (30) transparently show the effect of τ_L and τ_P respectively on the post-government income inequality.

A3. Proof of Proposition 2

Due to (A1) and (A3) of the proof of *Proposition 1*, at the social optimum,

$$\frac{dSV}{d\tau_L^*} = 0 = \frac{(1-\delta)}{\beta\delta}\frac{dSV}{d\tau_P^*} \tag{A8}$$

which means that

$$(1+\beta\delta)[\frac{1}{\alpha + \frac{\tau_P^*}{h^*} - 1} - \frac{1}{\alpha} - \frac{(1+\chi)}{(1+\eta)}\frac{1}{\{(1-\frac{\tau_P^*}{h^*}) + \frac{\beta\delta}{1-\delta}(1-\tau_P^*)\}}] + \frac{1-\delta}{1+\eta} =$$

$$(1+\beta\delta)\frac{(1-\delta)}{\beta\delta}[\frac{1}{\alpha + \tau_P^* - 1} - \frac{1}{\alpha} - \frac{(1+\chi)}{1+\eta}\frac{\frac{\beta\delta}{(1-\delta)}}{\{(1-\frac{\tau_P^*}{h^*}) + \frac{\beta\delta}{1-\delta}(1-\tau_P^*)\}}] + \frac{(1-\delta)}{\beta\delta}\frac{\beta\delta}{(1+\eta)}. \tag{A9}$$

Rearranging (A9) entails

$$(\alpha + \tau_P^* - 1) - \frac{(1-\delta)}{\beta\delta}(\alpha + \frac{\tau_P^*}{h^*} - 1) = \frac{(\beta\delta + \delta - 1)}{\alpha\beta\delta}(\frac{\tau_P^*}{h^*} + \alpha - 1)(\tau_P^* + \alpha - 1) \tag{A10}$$

which implies (31). At the same time, since $h^* \tau_L^* = \tau_P^*$, (A10) is restated in terms of τ_L^* as follows.

$$h^* = \frac{(1-\alpha)\frac{\beta\delta}{1-\delta} - 1 + (\alpha-1)(\frac{\beta\delta}{1-\delta} - 1)\frac{1}{\tau_L^*}}{(\frac{\beta\delta}{1-\delta} - 1)\tau_L^* + 1 - \alpha - \frac{\beta\delta}{1-\delta}} = \frac{\tau_P^*}{\tau_L^*} \tag{A11}$$

which is equivalent to (31).

On the other hand, by subtracting the common terms of both sides of (A9), we get

$$(1+\beta\delta)[\frac{1}{\alpha + \frac{\tau_P^*}{h^*} - 1} - \frac{1}{\alpha}] = (1+\beta\delta)\frac{(1-\delta)}{\beta\delta}[\frac{1}{\alpha + \tau_P^* - 1} - \frac{1}{\alpha}] \tag{A12}$$

which immediately implies (32).

A4. Proof of Proposition 3

According to the proofs of *Proposition 1 and 2*, at the social optimum, (A8) and thus (A12) are met. Furthermore, (A12) is restated as

$$\frac{\frac{\beta\delta}{1-\delta}(\alpha + \tau_P^* - 1) - (\alpha + \tau_L^* - 1)}{(\alpha + \tau_I^* - 1)(\alpha + \tau_P^* - 1)} = \left(\frac{\beta\delta}{1-\delta} - 1\right)\frac{1}{\alpha}. \tag{A13}$$

Rearranging (A13) entails

$$\tau_P^* - \tau_L^* = \frac{(1-\delta)}{\beta\delta}\left(1 - \frac{\beta\delta}{1-\delta}\right)\left(\alpha + \tau_L^* - 1\right)\frac{(1-\tau_P^*)}{\alpha}. \tag{A14}$$

Since $\frac{\delta}{1-\delta} \in (0,1)$, $\beta \in (0,1)$, (13), the sign of the right-hand side of (A14) is determined by the sign of $(\alpha + \tau_L^* - 1)$. In this regard, we can show that the sign of $(\alpha + \tau_L^* - 1)$ is positive by way of contradiction. Suppose that $(\alpha + \tau_L^* - 1) \leq 0$ at the social optimum. Then, the aggregate post-tax labor income at the social optimum is

$$\int_1^\infty \lambda_L^*(\theta l)^{1-\tau_L^*} dF_\theta = \frac{\alpha\lambda_L^*}{1-\alpha-\tau_L^*}\{(1-\tau_L^*) + \frac{\beta\delta}{1-\delta}(1-\tau_P^*)\}^{\frac{1-\tau_L^*}{1+\eta}}\{\lim_{\theta\to\infty}(\theta)^{1-\alpha-\tau_L^*} - 1\}. \tag{A15}$$

If $(\alpha + \tau_L^* - 1) \leq 0$, the last term of the right-hand side of (A15) goes to infinity or is zero so that the aggregate post-tax labor income cannot be defined or is none. Firstly, if the aggregate post-tax labor income goes to infinity, the competitive equilibrium of this economy and the social optimum cannot be defined in the first place. Secondly, if the aggregate post-tax labor income is zero, workers have no resources for their pre-retirement consumption, which derives the value of the social welfare into negative infinity, which is not the social optimum

either. Therefore, $(\alpha + \tau_L^* - 1) > 0$, which implies that the right-hand side of (A14) is strictly positive. Hence, $\tau_P^* > \tau_L^*$.

A5. Proof of Proposition 4

As the Gini index of unequally endowed earning ability and pre-government income is $\dfrac{1}{2\alpha - 1}$, whether the pre-government inequality positively affects the optimal degree of progressivity of public pension benefit or not is indicated by the sign of $-\dfrac{d\tau_P^*}{d\alpha}$. Applying the Implicit Function Theorem to (24) of *Proposition 1*,

$$-\frac{d\tau_P^*}{d\alpha} = \frac{(1+\beta\delta)\{\dfrac{-(2\alpha + \tau_P^* - 1)}{\alpha^2(\alpha + \tau_P^* - 1)^2}(1 - \tau_P^*)\}}{\dfrac{d^2SV}{d\tau_P^{*2}}}. \qquad (A16)$$

Due to (13), the sign of the numerator of (A16) is determined by the sign of $-(2\alpha + \tau_P^* - 1)$. In this regard, notice that $(\alpha + \tau_P^* - 1) > 0$ by the same logic for $(\alpha + \tau_L^* - 1) > 0$ as shown in the proof of *Proposition 3*. If not (i.e., if $(\alpha + \tau_P^* - 1) \leq 0$), the aggregate public pension benefit income goes to infinity or becomes zero to violate the government budget constraint of (7) at the social optimum. In turn, since $\alpha > 1$, $(\alpha + \tau_P^* - 1) > 0$, which implies that the numerator of (A16) is strictly negative. Moreover, the denominator of (A16) is strictly negative as well.

$$\frac{d^2SV}{d\tau_P^{*2}} = (1+\beta\delta)[\frac{-1}{(\alpha + \tau_P^* - 1)^2} - \frac{(1+\chi)}{(1+\eta)}\frac{(\dfrac{\beta\delta}{1-\delta})^2}{\{(1 - \tau_L^*) + \dfrac{\beta\delta}{1-\delta}(1 - \tau_P^*)\}^2}] < 0. \quad (A17)$$

Therefore, $-\dfrac{d\tau_P^*}{d\alpha} > 0$ which means that the pre-government inequality positively affects the optimal degree of progressivity of public pension benefit.

By the same token, whether the pre-government inequality positively affects the optimal degree of progressivity of labor income tax or not is indicated by the sign of $-\dfrac{d\tau_L^*}{d\alpha}$. Applying the Implicit Function Theorem to (27) of *Proposition 1*,

$$-\frac{d\tau_L^*}{d\alpha} = \frac{(1+\beta\delta)\{\dfrac{-(2\alpha + \tau_L^* - 1)}{\alpha^2(\alpha + \tau_L^* - 1)^2}(1 - \tau_L^*)\}}{(1+\beta\delta)[\dfrac{-1}{(\alpha + \tau_L^* - 1)^2} - \dfrac{(1+\chi)}{(1+\eta)}\dfrac{1}{\{(1 - \tau_L^*) + \dfrac{\beta\delta}{1-\delta}(1 - \tau_P^*)\}^2}]} > 0 \quad (A18)$$

due to (11), $\alpha > 1$ and $(\alpha + \tau_L^* - 1) > 0$. Thus, (A19) shows that the effect of the pre-government inequality on the optimal degree of labor income tax progressivity is positive.

A6. Proof of Proposition 5

To begin, whether the effect of population aging on the optimal degree of progressivity of public pension benefit is positive or negative is identified by the sign of $\dfrac{d\tau_P^*}{d\delta}$. Applying the Implicit Function Theorem to (24) of *Proposition 1*,

$$
\frac{d\tau_P^*}{d\delta} = -\frac{1}{\dfrac{d^2 SV}{d\tau_P^{*2}}} \left\{ \frac{\beta}{1+\eta} + \beta\left[\frac{(1-\tau_P^*)}{\alpha(\alpha+\tau_P^*-1)} - \frac{(1+\chi)}{(1+\eta)}\frac{\dfrac{\beta\delta}{(1-\delta)}}{\left\{(1-\tau_L^*)+\dfrac{\beta\delta}{1-\delta}(1-\tau_P^*)\right\}}\right] - \right.
$$
$$
\left. (1+\beta\delta)\frac{(1+\chi)}{(1+\eta)}\frac{\beta}{(1-\delta)^2}\frac{(1-\tau_L^*)}{\left\{(1-\tau_L^*)+\dfrac{\beta\delta}{1-\delta}(1-\tau_P^*)\right\}^2}\right\}.
$$
(A19)

Although it is certain that $\dfrac{d^2 SV}{d\tau_P^{*2}} < 0$, it is not certain whether $\dfrac{d\tau_P^*}{d\delta}$ is positive or negative, because the terms in the bracket of the right-hand side of (A20) take opposite signs whose sum is not certainly positive or negative. Hence, the effect of population aging on the optimal degree of progressivity of public pension benefit is ambiguous.

Moreover, whether the effect of population aging on the optimal degree of progressivity of labor income tax is positive or negative is identified by the sign of $\dfrac{d\tau_L^*}{d\delta}$. Applying the Implicit Function Theorem to (27) of *Proposition 1*,

$$
\frac{d\tau_L^*}{d\delta} = -\frac{1}{\dfrac{d^2 SV}{d\tau_L^{*2}}} \left\{ \frac{-1}{1+\eta} + \beta\left[\frac{(1-\tau_L^*)}{\alpha(\alpha+\tau_L^*-1)} - \frac{(1+\chi)}{(1+\eta)}\frac{1}{\left\{(1-\tau_L^*)+\dfrac{\beta\delta}{1-\delta}(1-\tau_P^*)\right\}}\right] + \right.
$$
$$
\left. (1+\beta\delta)\frac{(1+\chi)}{(1+\eta)}\frac{\beta}{(1-\delta)^2}\frac{(1-\tau_P^*)}{\left\{(1-\tau_L^*)+\dfrac{\beta\delta}{1-\delta}(1-\tau_P^*)\right\}^2}\right\}.
$$
(A20)

Although it is certain that $\dfrac{d^2 SV}{d\tau_L^{*2}} < 0$, it is not certain whether $\dfrac{d\tau_P^*}{d\delta}$ is positive or negative, because the terms in the bracket of the right-hand side of (A22) take opposite signs whose sum is not certainly determined as being positive or negative. Thus, the effect of population aging on the optimal degree of progressivity of labor income tax is ambiguous.

Appendix B

B1. Proof of Proposition 1, 2 and 3 with Lognormal Distribution of Earning Ability

Consider the model economy that is described in Section 2 now with different distribution of earning ability. In particular, earning ability $\theta \in [0, \infty)$ is distributed following a Lognormal distribution of $Lognormal(\mu, \sigma)$ with $\mu > 0$ and $\sigma > 0$. To reflect this change, optimal values under Lognormal distribution of earning ability are denoted with the superscript of $\lozenge$, instead of *. Note that the parameter of σ indicates pre-government inequality, as the Gini index of unequally endowed earning abilities is $2\Phi(\frac{\sigma}{\sqrt{2}}) - 1$ where $\Phi(\cdot)$ is the cumulative distribution function of the standard Normal distribution of $Normal(0,1)$, according to the formula of (A4). From (5) and (17), the competitive-equilibrium total output of this economy is

$$Y^\circ = (1 - \delta)\{(1 - \tau_L) + \frac{\beta\delta}{1 - \delta}(1 - \tau_P)\}^{\frac{1}{1+\eta}} exp(\mu + \frac{\sigma^2}{2}) \qquad (B1)$$

from which we can define $gY^\lozenge = G^\lozenge$. As in (17), $l^\circ(\theta) = \{(1 - \tau_L) + \frac{\beta\delta}{1 - \delta}(1 - \tau_P)\}^{\frac{1}{1+\eta}}$

for any given θ. Following the same steps of Section 2, the social welfare value function of $SV^{LN}(\tau_P, \tau_L, \rho)$ under Lognormal distribution of earning ability which satisfies all the conditions for being supported as a competitive equilibrium is

$$SV^{LN} = (1 + \beta\delta)[\tau_L\mu - \frac{\sigma^2}{2}(\tau_L^2 - 2\tau_L) + \tau_P\mu - \frac{\sigma^2}{2}(\tau_P^2 - 2\tau_P) + \frac{(1 + \chi)}{1 + \eta}log((1 - \tau_L) +$$

$$\frac{\beta\delta}{1-\delta}(1 - \tau_P)) + log(1 - \rho - \frac{g}{1-\delta}) + log(\rho)] - \frac{(1-\delta)}{1+\eta}\{(1 - \tau_L) + \frac{\beta\delta}{1-\delta}(1 - \tau_P)\} +$$

$$(1 + \beta\delta)\chi[log(g) + log(1 - \delta) + \mu + \frac{\sigma^2}{2}] + log(\frac{1-\delta}{\delta}) + \beta\delta log(\beta) + \delta log(\frac{\beta\delta}{1-\delta}) + \qquad (B2)$$

$$(\delta - 1) log(1 + r).$$

The optimal public pension system and labor income tax schedule (τ_P°, ρ°, λ_P°, τ_L° and λ_L°) are identified by maximizing $SV^{LN}(\tau_P, \tau_L, \rho)$, whose consequent allocation of individuals' private consumption and labor supply is implemented in decentralized way via competitive market. Since $SV^{LN}(\tau_P, \tau_L, \rho)$ is concave in τ_P, ρ and τ_L, the first-order condition is sufficient. First, the first-order condition for τ_P° defines the following formula of optimal progressivity of public pension benefit as

$$(1 + \beta\delta)[\sigma^2(1 - \tau_P^\circ) + \mu] = \frac{\beta\delta}{(1 - \delta)}[\frac{(1 + \beta\delta)\frac{(1 + \chi)}{(1 + \eta)}}{\{(1 - \tau_L^\circ) + \frac{\beta\delta}{1 - \delta}(1 - \tau_P^\circ)\}} - \frac{(1 - \delta)}{(1 + \eta)}]. \qquad (B3)$$

Like (24) of *Proposition 1* in Section 3, (B3) shows that the optimal degree of progressivity of public pension benefit is chosen to equate the social marginal benefit of equity

improvement from an increase in τ_P (the left-hand side) with its net social marginal cost of efficiency loss of labor-supply reduction (the right-hand side). Second, the first-order condition of $\dfrac{dSV^{LN}}{d\rho^\circ} = (1 + \beta\delta)\{\dfrac{1}{\rho^\circ} + \dfrac{-1}{1 - \rho^\circ - \dfrac{g}{1 - \delta}}\} = 0$ entails the optimal rate of public pension contribution as

$$\rho^\circ = \frac{1}{2}(1 - \frac{g}{1 - \delta}) \tag{B4}$$

which is isomorphic to (25) of *Proposition 1*. Third, satisfying the budget constraint of the government, (7), as well as (17) and (B1), the optimal value of λ_P° is

$$\lambda_P^\circ = \frac{\rho^\circ(1 - \delta)}{\delta}\{(1 - \tau_L^\circ) + \frac{\beta\delta}{1 - \delta}(1 - \tau_P^\circ)\}^{\frac{\tau_P^\circ}{1+\eta}} exp[\,\tau_P^\circ\mu - \frac{\sigma^2}{2}((\tau_P^\circ)^2 - 2\tau_P^\circ)]. \tag{B5}$$

Fourth, the first-order condition for τ_L° defines the following formula of optimal progressivity of income tax:

$$(1 + \beta\delta)[\sigma^2(1 - \tau_L^\circ) + \mu] = \frac{(1 + \beta\delta)\dfrac{(1 + \chi)}{(1 + \eta)}}{\left\{(1 - \tau_L^\circ) + \beta\dfrac{\delta}{1 - \delta}(1 - \tau_P^\circ)\right\}} - \frac{(1 - \delta)}{(1 + \eta)}. \tag{B6}$$

Fifth, satisfying the fiscal budget constraint of (6) as well as (17) and (B1), the optimal value of λ_L° is

$$\lambda_L^\circ = [1 - \rho^\circ - \frac{g}{(1 - \delta)}]\{(1 - \tau_L^\circ) + \frac{\beta\delta}{1 - \delta}(1 - \tau_P^\circ)\}^{\frac{\tau_L^\circ}{1+\eta}} exp[\,\tau_L^\circ\mu - \frac{\sigma^2}{2}((\tau_L^\circ)^2 - 2\tau_L^\circ)]. \tag{B7}$$

Like (27) of *Proposition 1* in Section 3, (B6) shows that the optimal degree of income tax progressivity is selected to equate the social marginal benefit of an increase in τ_L (the left-hand side) with its net marginal social cost of efficiency loss (the right-hand side). Therefore, *Proposition 1* is still valid with Lognormal distribution of earning ability.

In addition, corresponding to (29) and (30), based on the formula of (A4), for any given τ_P and τ_L, the Gini index of post-tax labor income of $\lambda_L(\theta l^\circ)^{1-\tau_L}$ is $2\Phi(\dfrac{\sigma(1 - \tau_L)}{\sqrt{2}}) - 1$ while that of the present value of public pension benefit income of $\dfrac{1}{1 + r}\lambda_P(\theta l^\circ)^{1-\tau_P}$ (the remaining part of the total post-government income) is $2\Phi(\dfrac{\sigma(1 - \tau_P)}{\sqrt{2}}) - 1$.

Now, to see whether *Proposition 2* still holds with Lognormal distribution of earning ability, notice that, at the social optimum, $\dfrac{dSV^{LN}}{d\tau_L^\circ} = 0 = \dfrac{(1 - \delta)}{\beta\delta}\dfrac{dSV^{LN}}{d\tau_P^\circ}$. From (B3) and (B6), we get

$$(1 + \beta\delta)[\sigma^2(1 - \frac{\tau_P^\circ}{h^\circ}) + \mu - \frac{(1 + \chi)}{(1 + \eta)}\frac{1}{\{(1 - \frac{\tau_P^\circ}{h^\circ}) + \frac{\beta\delta}{1 - \delta}(1 - \tau_P^\circ)\}}] + \frac{1 - \delta}{1 + \eta} =$$

$$(1 + \beta\delta)\frac{(1 - \delta)}{\beta\delta}[\sigma^2(1 - \tau_P^\circ) + \mu - \frac{(1 + \chi)}{1 + \eta}\frac{\frac{\beta\delta}{1 - \delta}}{\{\{(1 - \frac{\tau_P^\circ}{h^\circ}) + \frac{\beta\delta}{1 - \delta}(1 - \tau_P^\circ)\}\}}] + \qquad \text{(B8)}$$

$$\frac{(1 - \delta)}{\beta\delta}\frac{\beta\delta}{(1 + \eta)}.$$

Simplifying (B8) defines the optimal ratio of public pension benefit progressivity to income tax progressivity as

$$h^\circ = \frac{\frac{\beta\delta}{1 - \delta}\sigma^2\tau_P^\circ}{\sigma^2\tau_P^\circ - (1 - \frac{\beta\delta}{1 - \delta})(\sigma^2 + \mu)} = \frac{\tau_P^\circ}{\tau_L^\circ} \qquad \text{(B9)}$$

which depends on the chosen degree of progressivity, the pre-government inequality, and the elderly population share, as (31) of *Proposition 2* does. The logic underlying the optimal ratio of $h^\lozenge$ is the same as (32) of *Proposition 2* (no distortion on inter-temporal allocation) since by rearranging (B8) we obtain

$$\frac{\frac{\beta\delta}{1 - \delta}}{1} = \frac{(1 + \beta\delta)[\sigma^2(1 - \tau_P^\circ) + \mu]}{(1 + \beta\delta)[\sigma^2(1 - \frac{\tau_P^\circ}{h^\circ}) + \mu]} = \frac{(1 + \beta\delta)[\sigma^2(1 - \tau_P^\circ) + \mu]}{(1 + \beta\delta)[\sigma^2(1 - \tau_L^\circ) + \mu]} \qquad \text{(B10)}$$

which corresponds to (32) of *Proposition 2*. As shown in (B3), the numerator of the right-hand side of (B10) is the social marginal benefit of reducing the post-retirement income inequality by τ_P. As shown in (B6), the denominator of the right-hand side of (B10) is the social marginal benefit of reducing the pre-retirement income inequality by τ_L. Therefore, *Proposition 2* still holds with Lognormal distribution of earning ability.

Now, notice that simplifying (B8) ensues $\frac{\beta\delta}{1 - \delta}[\sigma^2(1 - \tau_L^\circ) + \mu] = [\sigma^2(1 - \tau_P^\circ) + \mu]$ which is restated as

$$\tau_P^\circ - \tau_L^\circ = \frac{1}{\sigma^2}(1 - \frac{\beta\delta}{1 - \delta})[\sigma^2(1 - \tau_L^\circ) + \mu]. \qquad \text{(B11)}$$

Since $\frac{\delta}{1 - \delta} \in (0,1)$, $\beta \in (0,1)$, $\mu > 0$, $\sigma > 0$ and (11), the sign of the right-hand side of (B11) is strictly positive. Therefore, $\tau_P^\circ > \tau_L^\circ$, proving *Proposition 3* with Lognormal distribution of earning ability.

B2. Proof of Proposition 4 and 5 with Lognormal Distribution of Earning Ability

Since the Gini index of unequally endowed earning ability and pre-government income is $2\Phi(\frac{\sigma}{\sqrt{2}}) - 1$, the pre-government inequality increases with σ. Thus, whether the pre-gov-

ernment inequality positively affects the optimal degree of progressivity of public pension benefit is indicated by the sign of $\dfrac{d\tau_P^\circ}{d\sigma}$. Applying the Implicit Function Theorem to (B3),

$$\frac{d\tau_P^\circ}{d\sigma} = -\frac{(1+\beta\delta)\{2\sigma(1-\tau_P^\circ)\}}{-(1+\beta\delta)\sigma^2} > 0 \tag{B12}$$

since $\delta \in (0,1)$, $\beta \in (0,1)$, $\sigma > 0$ and (13), showing that the effect of the pre-government inequality on the optimal degree of progressivity of public pension benefit is positive. To examine the effect of the pre-government inequality on the optimal progressivity of labor income tax, applying the Implicit Function Theorem to (B6) yields

$$\frac{d\tau_L^\circ}{d\sigma} = -\frac{(1+\beta\delta)\{2\sigma(1-\tau_L^\circ)\}}{-(1+\beta\delta)\sigma^2} > 0 \tag{B13}$$

since $\delta \in (0,1)$, $\beta \in (0,1)$, $\sigma > 0$ and (11), showing that the effect of the pre-government inequality on the optimal degree of progressivity of labor income tax is positive. Taking (B12) and (B13) together shows that *Proposition 4* still holds with Lognormal distribution of earning ability.

To identify the effect of population aging on the optimal degree of progressivity of public pension benefit $(\dfrac{d\tau_P^\circ}{d\delta})$, the Implicit Function Theorem is applied to (B3), yielding

$$\frac{d\tau_P^\circ}{d\delta} = \frac{\beta}{\sigma^2(1+\beta\delta)}\left[\sigma^2(1-\tau_P^\circ)+\mu - \frac{\dfrac{(1+\chi)}{(1+\eta)}\dfrac{\beta\delta}{(1-\delta)}}{\{(1-\tau_L^\circ)+\dfrac{\beta\delta}{1-\delta}(1-\tau_P^\circ)\}}\right] -$$

$$\frac{1}{\sigma^2}\frac{(1-\tau_L^\circ)\dfrac{(1+\chi)}{(1+\eta)}\dfrac{\beta}{(1-\delta)^2}}{\{(1-\tau_L^\circ)+\dfrac{\beta\delta}{1-\delta}(1-\tau_P^\circ)\}^2} + \frac{\beta}{\sigma^2(1+\beta\delta)(1+\eta)}. \tag{B14}$$

The terms in the right-hand side of (B15) take opposite signs, whose sum can be positive or negative. Hence, the effect of population aging on the optimal degree of progressivity of public pension benefit is ambiguous. To examine the effect of population aging on the optimal degree of labor income tax progressivity, applying the Implicit Function Theorem to (B6) yields

$$\frac{d\tau_L^\circ}{d\delta} = \frac{\beta}{\sigma^2(1+\beta\delta)}\left[\sigma^2(1-\tau_L^\circ)+\mu - \frac{\dfrac{(1+\chi)}{(1+\eta)}}{\{(1-\tau_L^\circ)+\dfrac{\beta\delta}{1-\delta}(1-\tau_P^\circ)\}}\right] -$$

$$\frac{1}{\sigma^2}\frac{(1-\tau_P^\circ)\dfrac{(1+\chi)}{(1+\eta)}\dfrac{\beta}{(1-\delta)^2}}{\{(1-\tau_L^\circ)+\dfrac{\beta\delta}{1-\delta}(1-\tau_P^\circ)\}^2} - \frac{1}{\sigma^2(1+\beta\delta)(1+\eta)}. \tag{B15}$$

With the terms in the right-hand side of (B17) taking opposite signs, it is not certainly determinable whether $\dfrac{d\tau_L^{\circ}}{d\delta}$ is positive or negative. Thus, the effect of population aging on the optimal progressivity of income tax is ambiguous. Taking (B14) and (B15) together proves that *Proposition 5* still holds with Lognormal distribution of earning ability.

Notes

1. In fact, even without public pension, it is impossible for Mirrlees model to consistently identify progressivity of the entire schedule of marginal income tax rates. As proven by Diamond (1998), optimal marginal income tax rate schedule from Mirrlees model is U-shaped. Thus, the slope of Mirrleesian optimal income tax schedule takes both negative and positive signs over different ranges of pre-tax incomes, which keeps us from consistently determining progressivity of the entire income tax schedule. In practice, actual marginal income tax rates monotonically increase with taxable income so that we can consistently determine the slope of the entire income tax schedule.

2. Notably, Chetty (2006) showed that log utility of consumption (the degree of relative risk aversion being one) is consistent with empirical findings on labor supply behavior.

3. According to Atkinson and Stiglitz (1976), since utility of consumption is separable from disutility of labor supply, as appears in (1), optimal capital income tax is zero for this economy. In line with this, according to Golosov *et al.* (2006), because there is no uncertainty on earning ability in the second period, zero capital income tax is optimal for this economy.

4. Using the quadratic formula, we could write a closed form of the optimal values of τ_P^* and τ_L^*, which are uniquely identifiable due to the feasible range of (11) and (13), like (25), (26) and (28). However, such a closed form presentation does not effectively show the underlying forces that shape the optimal degree of progressivity of public pension benefit and labor income tax as straightforward and informative as (24) or (27) does.

5. Using (31), taking a derivative of h^* with respect to α which indicates the effect of the pre-government inequality on the optimal ratio also involves $\dfrac{d\tau_P^*}{d\alpha}$ as well, which keeps us from clearly determining the sign of $\dfrac{dh^*}{d\alpha}$.

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Resumen

Este trabajo caracteriza la progresividad óptima de las pensiones de jubilación públicas y de los impuestos sobre la renta del trabajo, que conduce a redistribuciones intra e intergeneracionales. La progresividad óptima de las prestaciones de jubilación depende de la progresividad del impuesto sobre la renta salarial y viceversa. La relación óptima entre la progresividad de la prestación pública de jubilación y la progresividad del impuesto sobre la renta del trabajo no es igual a una tasa constante de descuento temporal. La pensión pública óptima es más progresiva que el impuesto sobre la renta del trabajo óptimo. Mientras que los efectos de la desigualdad antes de la intervención del gobierno sobre la progresividad óptima de la prestación pública de jubilación y la del impuesto sobre la renta del trabajo son positivos, el efecto del envejecimiento de la población es teóricamente ambiguo.

Palabras clave: progresividad de la prestación pública de jubilación, progresividad del impuesto sobre la renta.

Clasificación JEL: H55, H21, D60.